Some remarks on the dynamical systems approach to fourth order gravity

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Abstract. Building on earlier work, we discuss a general framework for exploring the cosmological dynamics of Higher Order Theories of Gravity. We show that once the theory of gravity has been specified, the cosmological equations can be written as a first-order autonomous system and we give several examples which illustrate the utility of our method. We also discuss a number of results which have appeared recently in the literature.

PACS numbers: 98.80.Jk, 04.50.+h, 05.45.-a

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1. Introduction

Although there are many good reasons to consider General Relativity (GR) as the best theory for the gravitational interaction, in the last few decades the advent of precision cosmology tests appears more and more to suggest that this theory may be incomplete. In fact, besides the well known problems of GR in explaining the astrophysical phenomenology (i.e. the galactic rotation curves and small scale structure formation), cosmological data indicates an underlying cosmic acceleration of the Universe which cannot be recast in the framework of GR without resorting to a additional exotic matter components. Several models have been proposed [1] in order to address this problem and currently the one which best fits all available observations (Supernovae Ia [2], Cosmic Microwave Background anisotropies [3], Large Scale Structure formation [4], baryon oscillations [5], weak lensing [6]), turns out to be the *Concordance Model* in which a tiny cosmological constant is present [7] and ordinary matter is dominated by a Cold Dark component. However, given that the Λ - CDM model is affected by significant fine-tuning problems related to the vacuum energy scale, it seems desirable to investigate other viable theoretical schemes.

It is for these reasons that in recent years many attempts have been made to generalize standard Einstein gravity. Among these models the so-called Extended Theory of Gravitation (ETG) and, in particular, *non-linear gravity theories* or *higher-order theories of gravity* have provided quite interesting results on both cosmological [8, 9, 10, 11] and astrophysical [10, 12] scales. These models are based on gravitational actions which are non-linear in the Ricci curvature R and/or contain terms involving combinations of derivatives of R [13, 14, 15]. The peculiarity of these models is related to the fact that field equations can be recast in such a way that the higher order corrections provide an energy - momentum tensor of geometrical origin describing an "effective" source term on the right hand side of the standard Einstein field equations [8, 9]. In this *Curvature Quintessence* scenario, the cosmic acceleration can be shown to result from such a new geometrical contribution to the cosmic energy density budget, due to higher order corrections to the HE Lagrangian.

Among higher order theories of gravity, fourth order gravity $f(R)$ models and in particular $f(R) = R^n$ [22] and $f(R) = R + 1/R$ [8, 9, 21] have recently gained particular attention since they seems to be able to provide an interesting alternative description of the cosmos [16]. Furthermore, these models can be related to other cosmologically viable models once the background dynamics has been introduced into the field equations [17], providing a possible theoretical explanation to some of them.

Because the field equations resulting from HTG are extremely complicated, the theory of dynamical systems provides a powerful scheme for investigating the physical behaviour of such theories (see for example [22, 23]). In fact, studying cosmologies using the dynamical systems approach has the advantage of providing a relatively simple method for obtaining exact solutions (even if these only represent the asymptotic behavior) and obtain a (qualitative) description of the global dynamics of these models. Consequently, such an analysis allows for an efficient preliminary investigation of these theories, suggesting what kind of models deserve further investigation.

In this paper, using the Dynamical Systems Approach (DSA) approach suggested by Collins and then by Ellis and Wainwright (see [24] for a wide class of cosmological models in the GR context), we develop a completely general scheme, which in principle allows one to analyze every fourth order gravity Lagrangian. Our study generalizes

[22], which considered a generic power law function of the Ricci scalar $f(R) = R^n$ and extends the general approach given in a recent paper [25]. Here a general analysis was obtained using a one-parameter description of any $f(R)$ model, which unfortunately turns out to be somewhat misleading.

The aim of this paper is to illustrate the general procedure for obtaining a phase space analysis for any analytical $f(R)$ Lagrangian, which is regular enough to be well defined up to the third derivative in R. After a short preliminary discussion about fourth order gravity, we will discuss this general procedure, giving particular attention to clarifying the differences between our approach and the one worked out in [25]. In order to illustrate these differences and the problems that exist in [25], we apply our method to two different families of Lagrangian $R^p \exp qR$ and $R + \alpha R^n$. The last part of the paper is devoted to discussion and conclusions.

2. Fourth Order Gravity Models

From a purely theoretical point of view there are no prescriptions which prevent one to describe the gravitational interaction using a Lagrangian that is non-linear in the Ricci scalar and/or contains combinations of the Ricci and Riemann tensor. The main argument that led us to choose what we call the Hilbert - Einstein Lagrangian is that only in this case does one obtain second order field equations in the metric and the Newtonian Poisson equation in the low energy limit. If we relax the assumption of linearity in the gravitational action, the general coordinate invariance allows, in principle, infinitely many additive terms to the HE action:

$$\mathcal{A}_G = \int d^4x\sqrt{-g}\left[\Lambda + c_0 R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + ...\right] \,, \tag{1}$$

and the general equations turn out to be particularly difficult to solve. However, if we limit ourselves to the fourth order, several simplifications are possible. First of all the Gauss-Bonnet theorem allows one to eliminate the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ terms. Moreover, in the case of homogeneous and isotropic spacetimes, the terms coming from the variation of the $R_{\mu\nu}R^{\mu\nu}$ invariant coincides with the one coming from the variation of the R^2 term. Finally, one can define a suitable parametrization which makes it possible to recast the higher order field equations as a system of second order differential equations together with the constraint given by the definition of the new variables [26].

Consequently, in cosmology, the "effective" fourth order Lagrangian can be considered a generic analytic function of the Ricci scalar $f(R)$ §:

$$L = \sqrt{-g}\left[f(R) + \mathcal{L}_M\right] \,. \tag{2}$$

By varying equation (2), we obtain the fourth order field equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = f'(R)^{;\alpha\beta}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right) + \tilde{T}^M_{\mu\nu} \,, \tag{3}$$

where $\tilde{T}^M_{\mu\nu} = \dfrac{2}{\sqrt{-g}}\dfrac{\delta(\sqrt{-g}L_M)}{\delta g_{\mu\nu}}$ and the prime denotes the derivative with respect to R. Standard Einstein equations are immediately recovered if $f(R) = R$. When $f'(R) \neq 0$ the equation (3) can be recast in the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{TOT}_{\mu\nu} = T^R_{\mu\nu} + T^M_{\mu\nu} \,, \tag{4}$$

§ Unless otherwise specified, we will use natural units ($\hbar = c = k_B = 8\pi G = 1$) and the $(+,-,-,-)$ signature.

where

$$T^{R}_{\mu\nu} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\mu\nu}\left[f(R) - Rf'(R)\right] + f'(R)^{;\alpha\beta}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu})\right\} , \qquad (5)$$

represent the stress energy tensor of an effective fluid sometimes referred to as the "curvature fluid" and

$$T^{M}_{\mu\nu} = \frac{1}{f'(R)}\tilde{T}^{M}_{\mu\nu} , \qquad (6)$$

represents an effective stress-energy tensor associated with standard matter.

The conservation properties of these effective fluids are given by the Bianchi identities $T_{\mu\nu}^{\ ;\nu}$. When applied to the total stress energy tensor, these identities reveal that if standard matter is conserved the total fluid is also conserved even though the curvature fluid may in general possess off–diagonal terms [22, 27, 28]. In other words, no matter how complicated the effective stress energy tensor $T^{TOT}_{\mu\nu}$ is, it will always be divergence free if $\tilde{T}^{M;\nu}_{\mu\nu} = 0$. When applied on the single effective tensors, the Bianchi identities read

$$T^{M;\nu}_{\mu\nu} = \frac{\tilde{T}^{M;\nu}_{\mu\nu}}{f'(R)} - \frac{f''(R)}{f'(R)^2}\,\tilde{T}^{M}_{\mu\nu}\,R^{;\nu} , \qquad (7)$$

$$T^{R;\nu}_{\mu\nu} = \frac{f''(R)}{f'(R)^2}\,\tilde{T}^{M}_{\mu\nu}\,R^{;\nu} , \qquad (8)$$

the last expression being a consequence of total energy-momentum conservation. It follows that the individual effective fluids are not conserved but exchange energy and momentum. It is worth noting that even if the *effective tensor* associated with the matter is not conserved, standard matter still follows the usual conservation equations $\tilde{T}^{M;\nu}_{\mu\nu} = 0$.

Let us now consider the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] . \qquad (9)$$

For this metric the action the field equations (5) reduce to

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{1}{3f'}\left\{\frac{1}{2}\left[f'R - f(R)\right] - 3H\dot{f}' + \mu_m\right\} , \\ 2\dot{H} + H^2 + \frac{k}{a^2} &= -\frac{1}{f'}\left\{\frac{1}{2}\left[f'R - f(R)\right] + \ddot{f}' - 3H\dot{f}' + p_m\right\} , \end{aligned} \qquad (10)$$

and

$$R = -6\left(2H^2 + \dot{H} + \frac{k}{a^2}\right) , \qquad (11)$$

where $H \equiv \dot{a}/a$, $f' \equiv \frac{df(R)}{dR}$ and the "dot" is the derivative with respect to t. The system (10) is closed by the only non trivial Bianchi identity for $\tilde{T}^{M}_{\mu\nu}$:

$$\dot{\mu}_m + 3H(\mu_m + p_m) = 0 , \qquad (12)$$

which corresponds to the energy conservation equation for standard matter.

3. The dynamical system approach in fourth order gravity theories

Following early attempts (see for example [29]), the first extensive analysis of cosmologies based on fourth order gravity theory using the DSA as defined in [24] was given in [22]. Here the phase space of the power law model $f(R) = \chi R^n$ was investigated in great detail, exact solutions were found and their stability determined. Following this, several authors have applied a similar approach to other types of Lagrangians [30], and very recently this scheme was generalized in [25].

In this paper we give a self consistent general technique that allows us to perform a dynamical system analysis of any analytic fourth order theory of gravity in the case of the FLRW spacetime.

The first step in the implementation of the Dynamical System Approach is the definition of the variables. Following [22], we introduce the general dimensionless variables :

$$\begin{aligned} &x = \frac{\dot{f}'}{f'H}, \qquad y = \frac{R}{6H^2}, \qquad z = \frac{f}{6f'H^2}, \\ &\Omega = \frac{\mu_m}{3f'H^2}, \qquad K = \frac{k}{a^2H^2}\ , \end{aligned} \tag{13}$$

where μ_m represents the energy density of a perfect fluid that might be present in the model ‖.

The cosmological equations (10) are equivalent to the autonomous system:

$$\begin{aligned} \frac{dx}{dN} &= \varepsilon\,(2K + 2z - x^2 + (K + y + 1)x) + \Omega\varepsilon\,(-3w - 1) + 2, \\ \frac{dy}{dN} &= y\varepsilon\,(2y + 2K + x\mathsf{q} + 4), \\ \frac{dz}{dN} &= z\varepsilon\,(2K - x + 2y + 4) + x\varepsilon\, y\mathsf{q}, \\ \frac{d\Omega}{dN} &= \Omega\varepsilon\,(2K - x + 2y - 3w + 1), \\ \frac{dK}{dN} &= K\varepsilon\,(2K + 2y + 2), \end{aligned} \tag{14}$$

where $N = |\ln a|$ is the logarithmic time and $\varepsilon = |H|/H$. In addition, we have the constraint equation

$$1 = -K - x - y + z + \Omega\,, \tag{15}$$

which can be used to reduce the dimension of the system. If one chooses to eliminate K, the variable associated with the spatial curvature, we obtain

$$\begin{aligned} \frac{dx}{dN} &= \varepsilon\,(4z - 2x^2 + (z - 2)x - 2y) + \Omega\varepsilon\,(x - 3w + 1), \\ \frac{dy}{dN} &= y\varepsilon\,[2\Omega + 2(z + 1) + x(\mathsf{q} - 2)], \end{aligned} \tag{16}$$

‖ In what follows we will consider only models containing a single fluid with a generic barotropic index. This might be problematic in treating the dust case because the condition $w = 0$ might lead to additional fixed points. This issue has been checked in our calculations and no change in the number of fixed points has been found. In addition, the generalization to a multi–fluid case is trivial: one has just to add a new variable Ω for each new type of fluid. This has the consequence of increasing the number of dynamical equations and therefore, the dimension of the phase space. However, since this generalization does not really add anything to the conceptual problem (at least in terms of a local analysis).

$$
\begin{aligned}
\frac{dz}{dN} &= z\varepsilon\,(2z+2\Omega-3x+2)z+x\varepsilon\, y\mathfrak{q},\\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon\,(2\Omega-3x+2z-3w-1),\\
K &= z+\Omega-x-y-1\,.
\end{aligned}
$$

The quantity $\mathfrak{q}$ is defined, in analogy with [25], as

$$\mathfrak{q} \equiv \frac{f'}{Rf''}\,. \tag{17}$$

The expression of $\mathfrak{q}$ in terms of the dynamical variables is the key to closing the system (29) and allows one to perform the analysis of the phase space. The crucial aspect to note here is that $\mathfrak{q}$ is a function of R only, so the problem of obtaining $\mathfrak{q}=\mathfrak{q}(x,y,z,\Omega)$ is reduced to the problem of writing $R=R(x,y,z,\Omega)$. This can be achieved by noting that the quantity

$$r \equiv -\frac{Rf'}{f} \tag{18}$$

is a function of R only and can be written as

$$r=-\frac{y}{z}\,. \tag{19}$$

Solving the above equation for R allows one to write R in terms of y and z and close the system (16).

In this way, once a Lagrangian has been chosen, we can in principle write the dynamical system associated with it using (16), substituting into it the appropriate form of $\mathfrak{q}=\mathfrak{q}(y,z)$. This procedure does however require particular attention. For example, there are forms of the function f for which the inversion of (19) is a highly non trivial (e.g. $f(R)=\cosh(R)$). In addition, the function $\mathfrak{q}$ could have a non-trivial domain, admit divergences or may not be in the class C^1, which makes the analysis of the phase space a very delicate problem. Finally, the number m of equations of (16) is always $m\geq 3$ and this implies that fourth order gravity models can admit chaotic behaviour. While this is not surprising, it makes the deduction of the non–local properties of the phase space a very difficult task.

The solutions associated with the fixed points can be found by substituting the coordinates of the fixed points into the system

$$\dot{H} = \alpha H^2\,, \qquad \alpha=-1-\Omega_i+x_i-z_i\,, \tag{20}$$

$$\dot{\mu}_m = \frac{3(1+w)}{\alpha\, t}\mu_m\,, \tag{21}$$

where the subscript "i" stands for the value of a generic quantity in a fixed point. This means that for $\alpha\neq 0$ the general solutions can be written as

$$a = a_0(t-t_0)^{1/\alpha}\,, \tag{22}$$

$$\mu_m = a_0(t-t_0)^{\frac{3(1+w)}{\alpha}}\,. \tag{23}$$

The expression above gives the solution for the scale factor and the evolution of the energy density for every fixed point in which $\alpha\neq 0$. When $\alpha=0$ the (20) reduces to $\dot{H}=0$ which correspond to either a static or a de Sitter solution.

The solutions obtained in this way have to be considered particular solutions of the cosmological equations in the same way that solutions can sometimes be found by using an ansatz for the form of the solution. For this reason it is important to stress

that only direct substitution of the results derived from this approach can ensure that the solution is physical (i.e. it satisfies the cosmological equations (10)). This check is also useful for understanding the nature of the solutions themselves e.g. to calculate the value of the integration constant(s).

Also, the fact that different fixed points correspond to the same solutions is due to the fact that at the fixed points the different terms in the equation combine in such a way to obtain the same evolution of the scale factor. This means that although two solutions are the same, the physical mechanism that realizes them can be different

In the following we will present a number of examples of $f(R)$ theories that can be analyzed with this method and we compare the results obtained with those given in [25] ¶. An analysis of the approach presented in [25] and the differences with our method are given in the Appendix.

4. Examples of $f(R)$ - Lagrangians

In this section we will show, with the help of some examples, how the DSA developed above can be applied. In particular we will consider the cases $f(R) = R^p \exp(qR)$ and $f(R) = R + \alpha R^n$. Since the aim of the paper is to provide only the general setting with which to develop the dynamical system approach in the framework of fourth order gravity, we will not give a detailed analysis of these models. A series of future papers will be dedicated to this task. In what follows, we will limit ourselves to the finite fixed points, their stability and the solutions associated with them. A comparison with the results of [25] will also be presented.

4.1. The $f(R) = R^p \exp(qR)$ case

Let us consider the Lagrangian $f(R) = R^p \exp(qR)$. As explained in the previous section, the dynamical system equations for this Lagrangian can be obtained by calculating the form of the parameter $\mathfrak{q}$. We have

$$\mathfrak{q}(y, z) = \frac{y\, z}{y^2 - p\, z^2}\,. \tag{24}$$

¶ One difference betwen our approach ad the one in [25] is that we consider a non zero spatial curvature k. This choice has been made with the aim of obtaining a completely general analysis of a fourth order cosmology from the dynamical systems point of view. In addition, since most of the observational values for the cosmological parameters are heavily model dependent, we chose to limit as much as possible the introduction of priors in the analysis. Anyway, the limit of flat spacelike sections ($K \to 0$) can be obtained in a straightforward way for our examples. In fact, each fixed point is associated with a specific value of the variable K (i.e. a value for k) and the stability of these points is independent from the value of K. This means that in order to consider the limit $K \to 0$ one has just to exclude the fixed points associated to $K \neq 0$. Also, looking at the dynamical equations one realizes that $K = 0$ is an invariant submanifold i.e. an orbit with initial condition $K = 0$ will not escape the subspace $K = 0$ and orbits with initial condition $K \neq 0$ can approach the hyperplane $K = 0$ only asymptotically. As a consequence, one does not need to have any other information on the rest of the phase space to characterize the evolution of the orbits in the submanifold $K = 0$.

Substituting this function into (29) we obtain

$$
\begin{aligned}
\frac{dx}{dN} &= \varepsilon\,[4z-2x^2+(z-2)x-2y]+\Omega\varepsilon\,(x-3w+1),\\
\frac{dy}{dN} &= y\varepsilon\,\left[2\Omega+2z+2+\frac{x\,z}{y^2-p\,z^2}-2x\right],\\
\frac{dz}{dN} &= z\varepsilon\,\left[2z+2\Omega-3x+2+\frac{x\,y}{y^2-p\,z^2}\right],\\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon\,(2\Omega-3x+2z-3w-1),\\
K &= z+\Omega-x-y-1\,.
\end{aligned}
\tag{25}
$$

The most striking feature of this system is the fact that two of the equations have a singularity in the hypersurface $y^2 = p\ z^2$. This, together with the existence of the invariant submanifolds $y = 0$ and $z = 0$ heavily constrains the dynamics of the system. In particular, it implies that no global attractor is present, so no general conclusion can be made on the behavior of the orbits without first providing information about the initial conditions. The finite fixed points can be obtained by setting the LHS of (25) to zero and solving for (x, y, z, Ω), the results are shown in Table 1.

Table 1. Fixed points of $R^p\exp(qR)$.The superscript "*" represents a point corresponding to a double solution.

Point	Coordinates (x, y, z, Ω)	K
$\mathcal{A}$	$(0,0,0,0)$	-1
$\mathcal{B}$	$(-1,0,0,0)$	0
$\mathcal{C}$	$(-1-3w,0,0,-1-3w)$	-1
$\mathcal{D}$	$(1-3w,0,0,2-3w)$	0
$\mathcal{E}$	$(2,0,2,0)$	-1
$\mathcal{F}^*$	$(1,-2,0,0)$	0
$\mathcal{G}$	$(0,-2,-1,0)$	0
$\mathcal{H}$	$(4,0,5,0)$	0
$\mathcal{I}$	$(2-2p,2p(1-p),2-2p,0)$	$2p(p-1)-1$
$\mathcal{L}^*$	$(-3(1+w),-2,0,-4-3w)$	0
$\mathcal{M}$	$\left(\frac{4-2p}{1-2p},\frac{(5-4p)p}{2p^2-3p+1},\frac{5-4p}{(p-1)(2p-1)},0\right)$	0
$\mathcal{N}$	$\left(\frac{-3(1+w)(p-1)}{p},\frac{3(1+w)-4p}{2p},\frac{-4p+3w+3}{2p^2},\frac{p(9w-2p(3w+4)+13)-3(w+1)}{2p^2}\right)$	0

The solutions corresponding to these fixed points can be obtained by substituting the coordinates into the system (20) and are shown in Table 2 [+].

The stability of the finite fixed points can be found using the Hartman-Grobman theorem [31]. The results are shown in Table 3. Note that some of the eigenvalues diverge for $p = 0, 1$. This happens because in the operations involved in the derivation of the stability terms $p-1$ and/or p appear in the denominators. However this is not a real pathology of the method but rather a consequence of the fact that for these two values of the parameter the cosmological equations assume a special form. In fact, as

[+] Note that even if the parameter q is not present in the dynamical equations it appears in the solutions because we have calculated the integration constants via direct substitution in the cosmological equations.

Table 2. Solutions associated with the fixed points of $R^p\exp(qR)$. The solutions are physical only in the intervals of p mentioned in the last column.

Point	Scale Factor	Energy Density	Physical
$\mathcal{A}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{B}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{C}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{D}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{E}$	$a(t)=(t-t_0)$	0	$p\geq 1$
$\mathcal{F}^*$	$\begin{cases} a(t)=a_0, \\ a(t)=a_0\exp\left[\pm\frac{\sqrt{2-3p}}{6\sqrt{q}}(t-t_0)\right], \end{cases}$	0	$p\geq 0$ $p<\frac{2}{3}, q>0 \vee p>\frac{2}{3}, q<0$
$\mathcal{G}$	$\begin{cases} a(t)=a_0, \\ a(t)=a_0\exp\left[\pm\frac{\sqrt{2-3p}}{6\sqrt{q}}(t-t_0)\right], \end{cases}$	0	$p\geq 0$ $p<\frac{2}{3}, q>0 \vee p>\frac{2}{3}, q<0$
$\mathcal{H}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$p\geq 2$
$\mathcal{I}$	$a(t)=(t-t_0)\sqrt{1-2p(p-1)}$	0	$1\leq p\leq\frac{1}{2}+\frac{\sqrt{3}}{2}$
$\mathcal{L}^*$	$\begin{cases} a(t)=a_0, \\ a(t)=a_0\exp\left[\pm\frac{\sqrt{2-3p}}{6\sqrt{q}}(t-t_0)\right], \end{cases}$	0	$p\geq 0$ $p<\frac{2}{3}, q>0 \vee p>\frac{2}{3}, q<0$
$\mathcal{M}$	$a(t)=a_0\,(t-t_0)^{\frac{2p^2-3p+1}{2-p}}$	$\mu_m=\mu_{m\,0}t^{\frac{3\left(2p^2-3p+1\right)(w+1)}{p-2}}$	$p=\frac{1}{2},1,\frac{5}{4}$
$\mathcal{N}$	$a(t)=a_0\,(t-t_0)^{\frac{2p}{3(w+1)}}$	$\mu_m=\mu_{m\,0}(t-t_0)^{-2p}$	$p=\frac{3(w+1)}{4}\quad(\mu_{m\,0}=0)$

we will see in the next section, if one starts the calculations using these critical values of p one ends up with eigenvalues that present no divergence.

Let us now compare our results with the ones in [25] (see the Appendix for details of this last method). The number of fixed points obtained for this Lagrangian, when $K=0$, matches the ones obtained in [25]. This result can be explained by the fact that the solutions of the constraint equation for m (A.1) coincide with the ones coming from the correct constraint equation (A.2) (the matching between the two systems can be obtained setting $w=0$ in Table 1). However, when one calculates the stability of these points our results are in strikingly different to those presented in [25]. For example, in our general formalism it turns out that the fixed point $\mathcal{M}$ is a saddle for any value of the parameter p and, as consequence, it can represent only a transient phase in the evolution of this class of models. Instead, in [25] the authors find that this point is a stable spiral and argue that this fact prevents the existence of cosmic histories in which a decelerated expansion is followed by an accelerated one. From this they also conclude that an entire subclass of these models ($m=m(p)>0$) can be ruled out. Our results show clearly that this is not the case. Another example relates to the point $\mathcal{N}$. In [25] the authors find that this point can be stable when $m\to 0$, but from Table 4 one finds that this point can only be a saddle. As explained in the Appendix, the reason behind these differences is the fact that the method used in [25] leads to incorrect results when, like in this case, there is no unambiguous way of determining the parameter $r=-y/z$ from the coordinates of the fixed points. Consequently the conclusions in [25] relating to the properties of these points are incorrect and have no physical meaning.

Table 3. The eigenvalues associated with the fixed points in $R^p\exp(qR)$. The eigenvalues of the fixed point $\mathcal{N}$ are displayed on three lines because of their mathematical complexity.

Point	Eigenvalues
$\mathcal{A}$	$[2,-2,2,-1-3w]$
$\mathcal{B}$	$[5,2,4,2-3w]$
$\mathcal{C}$	$[3(1+w),-2,2,1+3w]$
$\mathcal{D}$	$[3(1+w),2,4,-2+3w]$
$\mathcal{E}$	$[-2,-2,2,-2-3(1+w)]$
$\mathcal{F}$	$[-4,-2,0,-4-3w]$
$\mathcal{G}$	$\left[-2,-\frac{3}{2}-\frac{1}{2}\sqrt{\frac{68-25p}{p-4}},-\frac{3}{2}+\frac{1}{2}\sqrt{\frac{68-25p}{p-4}},-3(w+1)\right]$
$\mathcal{H}$	$[-5,4,2,-3(1+w)]$
$\mathcal{I}$	$\left[-2,-2+p-\sqrt{3\,p\,(3p-4)},-2+p+\sqrt{3\,p\,(3p-4)},2p-3(1+w)\right]$
$\mathcal{L}$	$[-4,-2,0,4+3w]$
$\mathcal{M}$	$\left[-4+\frac{1}{p-1},\frac{2(p-2)}{1-3p+p^2},-2+\frac{6}{1-2p}+\frac{2}{p-1},-4+\frac{3}{1-2p}+\frac{2}{p-1}-3w\right]$
$\mathcal{N}$	$\left[-1-\left\lvert\frac{p-3(\omega+1)}{p}\right\rvert,-1+\left\lvert\frac{p-3(\omega+1)}{p}\right\rvert,\ldots..\right.$
	$\ldots.\frac{3[(2p-1)w-1]}{4p}-\frac{1}{4}\sqrt{\frac{-81(1+w)+4p^2(8+3w)^2+3p(1+w)(139+87w)-4p^2(152+3w(55+18w))}{(p-1)p^2}},\ldots...$
	$\left.\ldots.\frac{3[(2p-1)w-1]}{4p}+\frac{1}{4}\sqrt{\frac{-81(1+w)+4p^2(8+3w)^2+3p(1+w)(139+87w)-4p^2(152+3w(55+18w))}{(p-1)p^2}}\right]$

4.2. The $f(R)=\exp(qR)$ case

Let us now consider the simple case of a pure exponential Lagrangian. Since this case has been extensively analyzed in a different paper [32], we sketch here only the main results which are interesting for our discussion, referring the reader to [32] for further details. The function $\mathfrak{q}$ is

$$\mathfrak{q}(y,z)\;=\;\frac{z}{y}\,, \tag{26}$$

and dynamical system equations read:

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon\left[4z-2y-x(2+2x-z-\Omega)\right]+\Omega\varepsilon\left(1-3w\right),\\
\frac{dy}{dN} &= 2y\varepsilon\left(2+2z+2\Omega-x\right),\\
\frac{dz}{dN} &= 2z\varepsilon\left(1+\Omega+z-x\right),\\
\frac{d\Omega}{dN} &= \Omega\varepsilon\left(2\Omega-1-3w+2z-3x\right),\\
K &= z+\Omega-x-y-1\,.
\end{aligned} \tag{27}$$

The coordinates of the fixed points, their eigenvalues and corresponding solutions are summarized in Table 5 .

The cosmology of the Lagrangian $f(R)\;=\;\exp(qR)$ shows two interesting de Sitter phases: the point $\mathcal{D}$ which is unstable and non hyperbolic point $\mathcal{C}$ that can behave as an attractor (see [32]). In addition it is possible to prove that there is a set of non zero

Table 4. The stability of the eigenvalues associated with the fixed points in the model $R^p\exp(qR)$. With the index $^+$ we have indicated the attractive nature of the spiral points.

Point	Stability
$\mathcal{A}$	saddle
$\mathcal{B}$	$\begin{cases} \text{repellor} & 0<w<2/3 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{C}$	saddle
$\mathcal{D}$	$\begin{cases} \text{repellor} & 2/3<w<1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{E}$	saddle
$\mathcal{F}$	saddle
$\mathcal{G}$	$\begin{cases} \text{attractor} & 0<w<1\cup 2<p\leq\frac{68}{25} \\ \text{spiral}^+ & 0\leq w\leq 1\cup\frac{68}{25}<p<4 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{H}$	saddle
$\mathcal{I}$	$\begin{cases} \text{attractor} & 0<w<1\cup\frac{1}{2}-\frac{\sqrt{3}}{2}<p\leq 0\vee\frac{4}{3}\leq p<\frac{1}{2}+\frac{\sqrt{3}}{2} \\ \text{spiral}^+ & 0\leq w\leq 1\cup 0<p<\frac{4}{3} \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{L}$	non hyperbolic
$\mathcal{M}$	$\begin{cases} \text{attractor} & 0<w<1\cup\ p<\frac{1}{2}(1-\sqrt{3})\vee\frac{1}{2}(1+\sqrt{3})<p<2 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{N}$	saddle

measure of initial conditions for which orbits connect these two points. In other words, such a Lagrangian can provide a natural framework both for inflation and the recent cosmic acceleration phenomenon. Nevertheless, it seems to lack an almost Friedmann phase which is required for structure formation.

Table 5. Coordinates of the fixed points,the eigenvalues, and solutions for $f(R)=\exp(qR)$. The superscript "*" represents indicates a double point

Point	Coordinates (x,y,z,Ω)	Eigenvalues	Solution
$\mathcal{A}$	$[0,0,0,0]$	$[-3w-1,-2,2,2]$	$a=a_o(t-t_o)$
$\mathcal{B}$	$[-1,0,0,0]$	$[2-3w,2,4,4]$	$a=a_o(t-t_o)^{\frac{1}{2}}$
$\mathcal{C}^*$	$[1,-2,0,0]$	$[-2,-4,-3w-4,0]$	$a=a_oe^{c(t-t_o)}$
$\mathcal{D}$	$[0,-2,-1,0]$	$[-\frac{\sqrt{17}+3}{2},\frac{\sqrt{17}-3}{2},-2,-3-3w]$	$a=a_oe^{c(t-t_o)}$
$\mathcal{E}^*$	$[1-3w,0,0,2-3w]$	$[3w-2,2,4,4]$	$a=a_o(t-t_o)^{\frac{1}{2}}$
$\mathcal{F}^*$	$[-3w-3,-2,0,-3w-4]$	$[3w+4,-2,-4,0]$	$a=a_oe^{c(t-t_o)}$
$\mathcal{G}$	$[-3w-1,0,0,-3w-1]$	$[3w+1,-2,2,2]$	$a=a_o(t-t_o)$

Table 6. The stability of the eigenvalues associated with the fixed points in the model $\exp(qR)$. See [32] for the stability of the non hyperbolic fixed points.

Point	Stability
$\mathcal{A}$	saddle
$\mathcal{B}$	$\begin{cases} \text{repellor} & 0 < w < 2/3 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{C}$	non hyperbolic
$\mathcal{D}$	saddle
$\mathcal{E}$	$\begin{cases} \text{repellor} & 2/3 < w < 1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{F}$	non hyperbolic
$\mathcal{G}$	saddle

If we now compare our results with ones described in [25], there are some clear differences. First, the number of fixed points turns out to be different. In fact, in our case there is no fixed point corresponding to the point P_4 of [25], and we obtain a new point $\mathcal{F}$ that does not appear in [25]. This follows directly from the pathological behaviour of equation (A.1). In fact, in the case of this Lagrangian, this expression and, in particular, the relation $m(r) = -r - 1$ has no solutions. Therefore, even in principle, there is no way to apply the method of [25] to this class of theories. Even if one refers to the correct equation (A.2), the only possibility of having $m(r)$ constant is to set $x = 0$, but this condition is not fulfilled by most of the fixed points. It is useful to see how these problems are related to the choice of taking m to be a parameter: if one substitutes the expression for m in terms of the dynamical variables into the initial dynamical system one obtains a set fixed points which correspond to the ones in Table 5.

Finally, differences arise also in the stability analysis. The two points $\mathcal{E}$ and $\mathcal{F}$ are non - hyperbolic and therefore require special treatment, without which it is impossible to draw general conclusions. Such treatment is given in detail in [32].

4.3. The case $f(R) = R + \alpha R^n$

Let us discuss now the case of a Lagrangian corresponding to a power law correction of the Hilbert - Einstein gravity Lagrangian $f(R) = R + \alpha R^n$. In this case, the characteristic function $\mathfrak{q}(r)$ reads:

$$\mathfrak{q}(r) = \frac{y}{n(z-y)}\,, \tag{28}$$

and substituting this relation into the system of equations (29) one obtains

$$
\begin{aligned}
\frac{dx}{dN} &= -2x^2 + (z-2)x - 2y + 4z + \Omega(x - 3w + 1), \\
\frac{dy}{dN} &= y\varepsilon\,[2\Omega + 2(z+1) + \frac{x\,y}{n(z-y)} - 2x], \\
\frac{dz}{dN} &= z\varepsilon\,(2z + 2\Omega - 3x + 2) + \varepsilon\,\frac{x\,y^2}{n(z-y)}, \\
\frac{d\Omega}{dN} &= \Omega\,\varepsilon\,(2\Omega - 3x + 2z - 3w - 1), \\
K &= z + \Omega - x - y - 1\,.
\end{aligned}
\tag{29}
$$

As in the case of $f(R) = R^p \exp(qR)$, the system is divergent on a hypersurface (this time $y = z$) but it admits only one invariant submanifold, namely $y = 0$ and $z = 0$. This, again, implies that no global attractor is present and no general conclusion can be made on the behavior of the orbits without giving information about the initial conditions. The finite fixed points, their eigenvalues, their stability and the solutions corresponding to them are summarized in Tables 7, 8, 9 and 10.

Table 7. Coordinate of the finite fixed points for $R + \alpha R^n$ gravity.

Point	Coordinates (x, y, z, Ω)	K
$\mathcal{A}$	$(0, 0, 0, 0)$	-1
$\mathcal{B}$	$(-1, 0, 0, 0)$	0
$\mathcal{C}$	$(-1-3w, 0, 0, -1-3w)$	-1
$\mathcal{D}$	$(1-3w, 0, 0, 2-3w)$	0
$\mathcal{E}$	$(0, -2, -1, 0)$	0
$\mathcal{F}$	$(2, 0, 2, 0)$	-1
$\mathcal{G}$	$(4, 0, 5, 0)$	0
$\mathcal{H}$	$(2(1-n), 2n(n-1), 2(1-n), 0)$	$2n(n-1)-1$
$\mathcal{I}$	$\left(\frac{2(n-2)}{2n-1}, \frac{(5-4n)n}{2n^2-3n+1}, \frac{5-4n}{2n^2-3n+1}, 0\right)$	0
$\mathcal{L}$	$\left(-\frac{3(n-1)(w+1)}{n}, \frac{-4n+3w+3}{2n}, \frac{-4n+3w+3}{2n^2}, \frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0

As before our results are different from those given in [25]. First of all, our set of fixed points do not coincide with the ones presented in [25]. In particular, in our analysis there is no fixed point corresponding to P_{5a}. Again, the reason for this difference is to be found in the constraint equation (A.1), which in this case gives the incorrect set of solutions and therefore affects the set of fixed points. In fact, if one substitutes the expression for $m(r)$ of [25] in terms of the coordinates in equations (34)-(39), it is easy to verify that two of these equations diverge at this point.

The differences between the results in our approach and the one presented in [25] are even more evident when the stability analysis is considered. For example, the point $\mathcal{E}$, corresponding to P_1, is always a saddle, except into the region $32/25 \le n < 2$ when it is attractive. This behavior is not obtained in [25] for which this point is stable only for $-2 < n < 0$. Also, points $\mathcal{G}$ (P_4) and $\mathcal{D}$ (P_3), which in our approach are always saddles (at least in the dust and in the radiation case), are always repellers in [25]. Finally, the stability of $\mathcal{I}$ corresponding to P_6 appears to be different from the one presented in [25].

Table 8. The eigenvalues associated with the fixed points in $R+\alpha R^n$.

Point	Eigenvalues
$\mathcal{A}$	$[-2,2,2,-3w-1]$
$\mathcal{B}$	$[5,4,2,2-3w]$
$\mathcal{C}$	$[-2,2,3w+1,3(w+1)]$
$\mathcal{D}$	$[4,2,3(w+1),3w-2]$
$\mathcal{E}$	$\left[-2,-\frac{3}{2}-\frac{1}{2}\sqrt{\frac{25n-32}{n}},-\frac{3}{2}+\frac{1}{2}\sqrt{\frac{25n-32}{n}},-3(w+1)\right]$
$\mathcal{F}$	$[-2,-2,2,-3(w+1)]$
$\mathcal{G}$	$[-5,4,2,-3(1+w)]$
$\mathcal{H}$	$\left[2(n-1),n-2-\sqrt{3n(3n-4)},n-2+\sqrt{3n(3n-4)},2n-3(w+1)\right]$
$\mathcal{I}$	$\left[\frac{3[(2n-1)w-1]}{4n}-\sqrt{\frac{-81(1+w)+4n^3(8+3w)^2+3n(1+w)(139+87w)-4n^2(152+3w(55+18w))}{16(n-1)n^2}},\right.$ $\frac{3[(2n-1)w-1]}{4n}+\sqrt{\frac{-81(1+w)+4n^3(8+3w)^2+3n(1+w)(139+87w)-4n^2(152+3w(55+18w))}{16(n-1)n^2}},$ $\left.\frac{1}{2}\left(3w-\frac{(5n+3(n-2)w-6)}{n}+1\right),\frac{1}{2}\left(3w+\frac{(5n+3(n-2)w-6)}{n}+1\right)\right]$
$\mathcal{L}$	$\left[-4+\frac{1}{n-1},-1+\frac{3}{-1+2n},-2+\frac{6}{-1+2n}+\frac{2}{-1+n},-4+\frac{3}{1-2n}+\frac{2}{-1+n}+\frac{2}{-1+n}-3w\right]$

5. Some remarks on the phase space of R^n-gravity

In the previous sections we have analyzed two cases of fourth order gravity Lagrangians, and the relative subcases, to illustrate how a general dynamical system approach can be formulated for these theories. Furthermore, we have discussed the differences between our approach and the one presented in [25]. In this section we compare the results of these methods when they are applied to $f(R)=\chi R^n$. The phase space of this class of theories has been investigated in detail in [22]. In the following we will show that only our method gives results that are in agreement with [22].

The crucial feature of R^n-gravity in terms of the general method discussed above is that the characteristic functions r and $\mathfrak{q}(r)$ are always constant. In particular, we have $r=-n$ $\mathfrak{q}(r)=n-1$. From the definition (19) it is then clear that the variables z and y are not independent, i.e., the phase space R^n-gravity is contained in the subspace $y=nz$ of the general phase space described by (29). This can be easily seen if one substitutes $y=nz$ into (29). Then the equations for y and z turn out to be exactly the same and (29) reduces to :

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon\left[-2x^2+\Omega x+zx-2x-2y+4z+\Omega(1-3w)\right] , \\
\frac{dy}{dN} &= y\varepsilon\left[2\Omega+\left(\frac{1}{n-1}-2\right)x+2(z+1)\right] , \\
\frac{d\Omega}{dN} &= \Omega\varepsilon\left[2\Omega-3x+2z-3w-1\right] ,
\end{aligned} \tag{30}$$

with the constraint

$$1+x-y-z+K=0 , \tag{31}$$

Table 9. The stability of the eigenvalues associated with the fixed points in the model $R + \alpha R^n$. The quantities A_i related to the fixed point $\mathcal{L}$, represent some non fractional numerical values ($A_1 \approx 1.220$, $A_1 \approx 1.224$, $A_3 \approx 1.470$).

Point	Stability
$\mathcal{A}$	saddle
$\mathcal{B}$	$\begin{cases} \text{repellor} & 0 < w < 2/3 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{C}$	saddle
$\mathcal{D}$	$\begin{cases} \text{repellor} & 2/3 < w < 1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{E}$	$\begin{cases} \text{attractor} & \frac{32}{25} \leq n < 2 \\ \text{spiral}^+ & 0 < n < \frac{32}{25} \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{F}$	saddle
$\mathcal{G}$	saddle
$\mathcal{H}$	$\begin{cases} \text{spiral}^+ & 0 < n < 1 \\ \text{saddle} & \text{otherwise} \end{cases}$
$\mathcal{I}$	$\begin{cases} w = 0, 1/3 & \text{saddle}, \\ w = 1 & \begin{cases} \text{repellor} & A_1 < n \leq A_2 \cup A_3 < n < \frac{3}{2}, \\ \text{saddle} & \text{otherwise} \end{cases} \end{cases}$
$\mathcal{L}$	$\begin{cases} w = 0 & \begin{cases} \text{repeller} & 1 < n < \frac{5}{4}, \\ \text{saddle} & \text{otherwise}, \end{cases} \\ w = 1/3 & \begin{cases} \text{attractor} & n < \frac{1}{2}(1-\sqrt{3}) \cup n > 2, \\ \text{repeller} & 1 < n < \frac{5}{4}, \\ \text{saddle} & \text{otherwise}, \end{cases} \\ w = 1 & \begin{cases} \text{repellor} & 1 < n < \frac{1}{14}(11+\sqrt{37}), \\ \text{saddle} & \text{otherwise} \end{cases} \end{cases}$

which is equivalent to the one given in [22]. Consequently the results obtained from the method presented above and [22] are identical *.

The same cannot be said for the results of [25]. In fact, although the set of fixed points are in agreement with the ones given in [22], the stability analysis is remarkably different. For example, the fixed point $\mathcal{G}$ is claimed to become a stable spiral for $n > 1$, preventing the presence of orbits with transient almost–Friedmann behavior. According to our results this is clearly not true since $\mathcal{G}$ is always a saddle-focus or a saddle in such an interval of n. Furthermore, $\mathcal{G}$ remains a saddle also for every $n \leq 0.33$ whereas in [25] is presented as a repeller for $n \to -1^-$. Other differences relate to the fixed point $\mathcal{B}$, which is never stable in our approach, but is suggested to be attractive for $3/4 < n < 1$ in [25].

* One can obtain a completely analogous result if the whole equations system (29) is considered without lowering the order of the equations. Of course one has to be careful in discarding the fixed points which do not fulfill the constraint $y = nz$.

Table 10. Solutions associated to the fixed points of $R+\alpha R^n$. The solutions are physical only in the intervals of p mentioned in the last column.

Point	Scale Factor	Energy Density	Physical
$\mathcal{A}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{B}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{C}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{D}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{E}^*$	$\begin{cases} a(t)=a_0, \\ a(t)=a_0\exp\left[\pm 2\sqrt{3}\alpha^\gamma(2-3n)^\gamma(t-t_0)\right], \end{cases}$ $\gamma=\frac{1}{2(1-n)}$	0	$n\geq 0$ $n<\frac{2}{3},\alpha>0\ \vee$ $n>\frac{2}{3},\alpha<0$
$\mathcal{F}$	$a(t)=(t-t_0)$	0	$n\geq 1$
$\mathcal{G}$	$a(t)=a_0\,(t-t_0)^{1/2}$	0	$n\geq 1$
$\mathcal{H}$	$a(t)=\sqrt{1-2n(n-1)}\,(t-t_0)$	0	$1\leq n\geq \frac{1}{2}+\frac{\sqrt{3}}{2}$
$\mathcal{I}^*$	$a(t)=a_0\,(t-t_0)^{\frac{2n^2-3n+1}{2-n}}$	$\mu_m=\mu_{m\,0}t^{-\frac{3\left(2n^2-3n+1\right)(w+1)}{n-2}}$	$n=\frac{1}{2},\mu_{m\,,0}=0$
$\mathcal{L}$	$a(t)=a_0\,(t-t_0)^{\frac{2n}{3(w+1)}}$	$\mu_m=\mu_{m\,,0}(t-t_0)^{2p}$	non physical

Table 11. Coordinates of the fixed points for the model $f(R)=\chi R^n$. The superscript "*" represents a double solution. The point $\mathcal{B}$ is a double solution for $n=0,2$.

Point	Coordinates (x,y,z,Ω)	K
$\mathcal{A}^*$	$(0,0,0,0)$	-1
$\mathcal{B}$	$(-1,0,0,0)$	0
$\mathcal{C}$	$(-1-3w,0,0,-1-3w)$	-1
$\mathcal{D}$	$(1-3w,0,0,2-3w)$	0
$\mathcal{E}$	$(2(1-n),-2n(n-1),2(1-n),0)$	$2n(n-1)-1$
$\mathcal{F}$	$\left(-\frac{3(n-1)(w+1)}{n},\frac{-4n+3w+3}{2n},\frac{-4n+3w+3}{2n^2},\frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0
$\mathcal{G}$	$\left(\frac{2(n-2)}{2n-1},\frac{(5-4n)n}{2n^2-3n+1},\frac{5-4n}{2n^2-3n+1},0\right)$	0

Table 12. Coordinates of the correct fixed points for the model $f(R)=\chi R^n$.

Point	Coordinates (x,y,z,Ω)	K
$\mathcal{A}$	$(0,0,0,0)$	-1
$\mathcal{B}$	$(-1,0,0,0)$	0
$\mathcal{C}$	$(-1-3w,0,0,-1-3w)$	-1
$\mathcal{D}$	$(1-3w,0,0,2-3w)$	0
$\mathcal{E}$	$(2(1-n),-2n(n-1),2(1-n),0)$	$2n(n-1)-1$
$\mathcal{F}$	$\left(-\frac{3(n-1)(w+1)}{n},\frac{-4n+3w+3}{2n},\frac{-4n+3w+3}{2n^2},\frac{-2(3w+4)n^2+(9w+13)n-3(w+1)}{2n^2}\right)$	0
$\mathcal{G}$	$\left(\frac{2(n-2)}{2n-1},\frac{(5-4n)n}{2n^2-3n+1},\frac{5-4n}{2n^2-3n+1},0\right)$	0

Table 13. Stability of the fixed points for R^n-gravity with matter. We consider here only dust or radiation, see [22] for details and the case of stiff matter. The term "spiral$^+$" has been used for pure attractive focus-nodes and the term "saddle-focus" for unstable focus-nodes.

	$n<\frac{1}{2}(1-\sqrt{3})$	$\frac{1}{2}(1-\sqrt{3})<n<0$	$0<n<1/2$	$1/2<n<1$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	repulsive	repulsive	repulsive	repulsive
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{E}$	saddle	attractive	spiral	spiral
$\mathcal{F}$	attractive	saddle	saddle	attractive

	$1<n<5/4$	$5/4<n<4/3$	$4/3<n<\frac{1}{2}(1+\sqrt{3})$	$n>\frac{1}{2}(1+\sqrt{3})$
$\mathcal{A}$	saddle	saddle	saddle	saddle
$\mathcal{B}$	saddle	repulsive	repulsive	repulsive
$\mathcal{C}$	saddle	saddle	saddle	saddle
$\mathcal{D}$	saddle	saddle	saddle	saddle
$\mathcal{H}$	spiral	spiral	attractive	saddle
$\mathcal{L}$	repulsive	saddle	saddle	attractive

$\mathcal{G}$	$n\lesssim 0.33$	$0.33\lesssim n\lesssim 0.35$	$0.35\lesssim n\lesssim 0.37$	$0.37\lesssim n\lesssim 0.71$	$0.71\lesssim n\lesssim 1$
$w=0$	saddle	saddle-focus	saddle-focus	saddle-focus	saddle
$w=1/3$	saddle	saddle	saddle-focus	saddle-focus	saddle-focus

	$1\lesssim n\lesssim 1.220$	$1.220\lesssim n\lesssim 1.223$	$1.223\lesssim n\lesssim 1.224$	$1.224\lesssim n\lesssim 1.28$
$w=0$	saddle-focus	saddle-focus	saddle-focus	saddle-focus
$w=1/3$	saddle-focus	saddle-focus	saddle-focus	saddle-focus

	$1.28\lesssim n\lesssim 1.32$	$1.32\lesssim n\lesssim 1.47$	$1.47\lesssim n\lesssim 1.50$	$n\gtrsim 1.50$
$w=0$	saddle-focus	saddle	saddle	saddle
$w=1/3$	saddle	saddle	saddle	saddle

6. Conclusions

In this paper we have presented a general formalism that allows one to apply DSA to a generic fourth order Lagrangian. The crucial point of this method is to express the two characteristic functions [25]:

$$\mathfrak{q}=\frac{f'}{Rf''}\,,\quad r\;=\;-\frac{Rf'}{f} \tag{32}$$

in terms of the dynamical variables, which, in principle, allows one to obtain a closed autonomous system for any Lagrangian density $f(R)$.

The resulting general system admits many interesting features, but is very difficult to analyze without specifying the function $\mathfrak{q}$ (i.e. the form of $f(R)$). Consequently, a "one parameter" approach can lead to a number of misleading results.

Even after substituting for q, the dynamical system analysis is still very delicate; in fact, q could be discontinuous, admit singularities or generate additional invariant submanifolds that influence deeply the stability of the fixed points as well as the global evolution of the orbits.

After describing the method, we applied it to two classes of fourth order gravity models: $R + \alpha R^n$ and $R^p \exp(qR)$, finding some very interesting preliminary results for the finite phase space. Both these models have fixed points with corresponding solutions that admit accelerated expansion and, consequently can be considered as possible candidates able to model either inflation or dark energy eras (or both). In addition, there are other fixed points which are linked to phases of decelerated expansion which can in principle allow for structure formation. These latter solutions are not physical for every value of their parameters, but this is not necessarily a problem. In fact, in order to obtain a Friedmann cosmology evolving towards a dark energy era, these points are required to be unstable i.e. cosmic histories coast past them for a period which depends on the initial conditions. This means that the general integral of the cosmological equations corresponding to such an orbit will only approximate the fixed point solution and this approximate behavior might still allow structures to form.

It is also important to mention the fact that even if one has the desired fixed points and desired stability, this does not necessarily imply that there is an orbit connecting them. This is due to the presence of singular and invariant submanifolds that effectively divide the phase space into independent sectors. Of course one can implement further constraints on the parameters in order to have all the interesting points in a single connected sector, but this is still not sufficient to guarantee that an orbit would connect them. The situation is made worse by the fact that, since the phase space is of dimension higher than three, chaotic behavior can also occur. It is clear then, that any statement on the global behavior of the orbits is only reliable if an accurate numerical analysis is performed. However, these issues (and others) will be investigated in more detail in a series of forthcoming papers.

A final comment is needed regarding the differences between our results and the ones given in [25]. Even if the introduction of q and r, was suggested for the first time in that paper, the results above (and in particular the existence of a viable matter era) are in disagreement with the ones given in that paper. The reason is that the authors of [25] used "a one parameter description" in order to deal with (29) in general. We were able to prove that, unfortunately, not only are the equations given in [25] incomplete, but also that the method also gives both incorrect and misleading conclusions.

Appendix A. The approach of [25]

The basic idea for closing the general system of autonomous equations for $f(R)$-gravity was suggested for the first time in [25]. In fact, if we define $m(r) = \mathsf{q}^{-1}$ the equations (29) for $w = 0$ and $K = 0$ are equivalent to the ones given in this paper. The authors of [25] proposed that the function m could be used as a parameter associated with the choice of $f(R)$, thus obtaining a "one parameter approach" to the dynamical systems analysis of $f(R)$ gravity. Unfortunately their method has several problems that lead to incorrect results. These problems can be avoided only if one considers the framework presented above.

Let us look at this issue in more detail ♯. In [25] the system equivalent to (29) is associated with the relation

$$\frac{dr}{dN} = r(1 + m(r) + r)\frac{\dot{R}}{HR}\,, \tag{A.1}$$

which is clearly a combination of the equations for z and y. In order to ensure that the variable r and consequently the parameter m is constant they require the RHS of the above equation to be zero. Their solution to this problem is the condition $1 + m(r) + r = 0$, which is an equation for r when the function $m(r)$ has been substituted for and is also the bases of their method of analysis.

The problem here is that this equation has not been fully expressed in terms of the dynamical system variables. In fact, one can rewrite (A.1) in the form:

$$\frac{dr}{dN} = \frac{r(1 + m(r) + r)}{m(r)}x\,, \tag{A.2}$$

which means that the condition $\frac{dr}{dN} = 0$ in fact corresponds to

$$\frac{r(1 + m(r) + r)}{m(r)}x = 0 \tag{A.3}$$

rather than $1 + m(r) + r = 0$. Equation (A.3) has a solution if

$$\begin{aligned} &x = 0, \\ &r = 0, \\ &\frac{(1 + m(r) + r)}{m(r)} \;=\; 0\,, \end{aligned} \tag{A.4}$$

and this leads to solutions for r which are in general different from the values of r obtained from $1 + m(r) + r = 0$. This inconsistency has major consequences for the rest of the analysis in [25], leading to changes in the number of fixed points as well as their stability (see the text above for details).

In fact, a more careful analysis reveals that for some of the fixed points (e.g. $P_1, ...P_4$) the values of r obtained from the relation $r = -y/z$ either cannot be determined unambiguously or do not solve the condition $1 + m(r) + r = 0$, which is claimed to come from (A.1) in [25].

This is a clear indication that the approach used in [25] is both incomplete and leads to wrong conclusions. It is also interesting to stress that if one substitutes the expression for m in terms of the dynamical system variables in (26-29) of [25], the results match the one obtained in our formalism. This implies that the reason the method described in [25] fails has its roots in the attempt to describe the phase space of a whole class of fourth order theories of gravity with only one parameter.

♯ It is important to note that in [25] the signature is not the same of the one used here (e.g -,+,+,+ instead of +,-,-,-) and the definition of the variables are slightly different. The transformation from one variable to another is as follows:

$$x \to -x_1, \quad y \to -x_3, \quad z \to x_2, \quad K \to 0, \quad w \to 0.$$

However, as expected, this does not affect our conclusions.

Appendix B. Acknowledgments

This work was supported by the National Research Foundation (South Africa) and the Ministrero degli Affari Esteri-DG per la Promozione e Cooperazione Culturale (Italy) under the joint Italy/South Africa science and technology agreement. A.T. warmly thanks the Department of Mathematics and Applied Mathematics of the Cape Town University for hospitality and for support.

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